# Geometric Issue and Sector Selection for Performance Attribution

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**Abstract**

This paper represents an overview of the fully geometric approach to performance attribution developed by Menchero of Vestek. In addition to providing further insights into the subtleties involved with geometric attribution, two new fully geometric approaches to attributions are provided. The first represents a slight extension of that provided by Menchero. The second constitutes a slight extension of the exponential approach developed by Carino of the Frank Russell Company. The advantages of the extended exponential approach include absolute transparency in interpretation of the results as well as a direct parallel to the algebraic aproach.

# 1 Overview of Geometric Performance Attribution

This Section contains a thorough overview of the fully geometric approach to attribution presented in [1]. In addition to the overview, additional insights are provided into the subtleties involved in the development of the formalism that will aid in the development of two new fully geometric approaches that are presented in the Sections that follow.

## 1.1 Arithmetic Relative Performance

As stated in [1], the arithmetic relative performance $\mathbf{P}^A_{iP}$ of a portfolio $P$ over the $i$th period is defined as the difference

$$\mathbf{P}^A_{iP} := R_{iP} - \overline{R}_{iP}, \tag{1}$$

where $R_{iP}$ is the portfolio return and $\overline{R}_{iP}$ is a benchmark return for the $i$th period. The first step in the evaluation is to partition the portfolio into $N_P$ weighted sectors with

$$R_{iP} = \sum_{j=1}^{N_P} w_{ij} R_{ij} \tag{2}$$

and

$$\overline{R}_{iP} = \sum_{j=1}^{N_P} \overline{w}_{ij} \overline{R}_{ij}, \tag{3}$$

where $w_{ij}$, $\overline{w}_{ij}$ are the weights and $R_{ij}$, $\overline{R}_{ij}$ are the returns for the $j$th sectors of the portfolio and benchmark respectively during the $i$th period.

The arithmetic relative performance for the $i$th period may then be further decomposed into aggregate arithmetic issue and sector selection performances via

$$\mathbf{P}^A_{iP} = \mathbf{I}^A_{iP} + \mathbf{S}^A_{iP}, \tag{4}$$

where

$$\mathbf{I}^A_{iP} = \sum_{j=1}^{N_P} \mathbf{I}^A_{ij} \tag{5}$$

is the aggregate arithmetic issue selection performance,

$$\mathbf{I}^A_{ij} = w_{ij}(R_{ij} - \overline{R}_{ij}) \tag{6}$$

is the arithmetic issue selection performance,

$$\mathbf{S}^A_{iP} = \sum_{j=1}^{N_P} \mathbf{S}^A_{ij}, \tag{7}$$

is the aggregate sector selection performance, and

$$\mathbf{S}^A_{ij} = (w_{ij} - \overline{w}_{ij})(\overline{R}_{ij} - \overline{R}_{iP}) \tag{8}$$

is the arithmetic sector selection performance. Note that

$$\mathbf{I}^A_{iP} = R_{iP} - \widetilde{R}_{iP} \tag{9}$$

and

$$\mathbf{S}^A_{iP} = \widetilde{R}_{iP} - \overline{R}_{iP}, \tag{10}$$

where

$$\widetilde{R}_{iP} := \sum_{j=1}^{N_P} w_{ij} \overline{R}_{iP} \tag{11}$$

is referred to as the semi-notional return.

The decomposition of the arithmetic relative performance into issue and sector selections allows for separate evaluation of the portfolio manager's choice of securities within each sector as well as their choice of weights for each sector. Although the separate issue and sector selection performances aid in the overall evaluation of the portfolio performance, there is no straightforward way to link the performance across multiple periods. This motivates geometric performance attribution, which has the advantage of being linkable across multiple periods in a straightforward manner as illustrated in the following Section.

## 1.2 Geometric Relative Performance

The geometric relative performance $\mathbf{P}^G_{iP}$ of a portfolio $P$ over the $i$th period is defined as

$$\mathbf{P}^G_{iP} := \frac{1 + R_{iP}}{1 + \overline{R}_{iP}}. \tag{12}$$

A distinct advantage of the geometric relative performance over the arithmetic relative performance is that it is trivial to link the performance of the portfolio across multiple periods. The total geometric relative performance $\mathbf{P}^G_{TP}$ over $N_T$ periods is simply the product of the geometric relative performances for each period, i.e.

$$\mathbf{P}^G_{TP} = \prod_{i=1}^{N_T} \mathbf{P}^G_{iP}. \tag{13}$$

This follows immediately from the fact that

$$1 + R_{TP} = \prod_{i=1}^{N_T} (1 + R_{iP}), \tag{14}$$

$$1 + \overline{R}_{TP} = \prod_{i=1}^{N_T} (1 + \overline{R}_{iP}), \tag{15}$$

where $R_{TP}$ and $\overline{R}_{TP}$ are the total returns for the portfolio and benchmark, respectively, and from

$$\mathbf{P}^G_{TP} = \frac{1 + R_{TP}}{1 + \overline{R}_{TP}}. \tag{16}$$

In analogy with arithmetic performance attribution, it is then desirable to also split the geometric relative performance into geometric issue and sector selection performances so that

$$\mathbf{P}^G_{TP} = (1 + \mathbf{I}^G_{TP})(1 + \mathbf{S}^G_{TP}), \tag{17}$$

where

$$1 + \mathbf{I}^G_{TP} = \frac{1 + R_{TP}}{1 + \widetilde{R}_{TP}} \tag{18}$$

is the geometric aggregate issue selection,

$$1 + \mathbf{S}^G_{TP} = \frac{1 + \widetilde{R}_{TP}}{1 + \overline{R}_{TP}} \tag{19}$$

is the geometric aggregate sector selection, and

$$1 + \widetilde{R}_{TP} = \prod_{i=1}^{N_T} (1 + \widetilde{R}_{iP}) \tag{20}$$

is the aggregate semi-notional return.

It is then desirable to further factor the geometric issue and sector selection performances into sectors via

$$1 + \mathbf{I}^G_{TP} = \prod_{i=1}^{N_T} (1 + \mathbf{I}^G_{iP}) = \prod_{i=1}^{N_T} \prod_{j=1}^{N_P} (1 + \mathbf{I}^G_{ij}) \tag{21}$$

and

$$1 + \mathbf{S}^G_{TP} = \prod_{i=1}^{N_T} (1 + \mathbf{S}^G_{iP}) = \prod_{i=1}^{N_T} \prod_{j=1}^{N_P} (1 + \mathbf{S}^G_{ij}). \tag{22}$$

If it were possible to perform this last facorization, then the total geometric relative performance over $N_T$ periods could be written as

$$\begin{aligned} \mathbf{P}^G_{TP} &= (1 + \mathbf{I}^G_{TP})(1 + \mathbf{S}^G_{TP}) \\ &= \prod_{i=1}^{N_T} (1 + \mathbf{I}^G_{iP})(1 + \mathbf{S}^G_{iP}) \\ &= \prod_{i=1}^{N_T} \prod_{j=1}^{N_P} (1 + \mathbf{I}^G_{ij})(1 + \mathbf{S}^G_{ij}). \end{aligned} \tag{23}$$

This would provide geometric linkage across both time and across sectors. Note also that this would imply a very localized geometric relative performance $\mathbf{P}^G_{ij}$ assigned to the $j$th sector during the $i$th period, i.e.

$$\mathbf{P}^G_{TP} = \prod_{i=1}^{N_T} \mathbf{P}^G_{iP} = \prod_{i=1}^{N_T} \prod_{j=1}^{N_P} \mathbf{P}^G_{ij}, \tag{24}$$

where

$$\mathbf{P}^G_{ij} = (1 + \mathbf{I}^G_{ij})(1 + \mathbf{S}^G_{ij}). \tag{25}$$

Equation (24) represents a fully geometric relative performance in that it is geometric across multiple periods as well as being geometric across sectors. To illustrate the meaning of the fully geometric approach, note that the order of the products in Equation (24) may be reversed so that you can define a geometric relative performance $\mathbf{P}^G_{Tj}$ for the $j$th sector over the entire $N_T$ periods via

$$\mathbf{P}^G_{Tj} := \prod_{i=1}^{N_T} \mathbf{P}^G_{ij} \tag{26}$$

so that

$$\mathbf{P}^G_{TP} = \prod_{i=1}^{N_T} \mathbf{P}^G_{iP} = \prod_{j=1}^{N_P} \mathbf{P}^G_{Tj}. \tag{27}$$

That is, you may first determine the geometric relative performance $\mathbf{P}^G_{iP}$ of the portfolio for each period and then determine the total geometric relative performance $\mathbf{P}^G_{TP}$ by taking the products of each, or you may first find the geometric relative performance $\mathbf{P}^G_{Tj}$ for each individual sector over all $N_T$ periods and then find the total relative geometric performance $\mathbf{P}^G_{TP}$ by taking their products. In other words, the resulting fully geometric performance attribution is geometric in both cross section and in time.

To gain further insight into the subtleties involved with deriving an explicit expression for Equation (24), consider the total return $R_{TP}$ for the full $N_T$ periods and $N_P$ sectors as given in Equation (14). The right hand side of Equation (14) may be subdivided into sectors via

$$\begin{aligned} 1 + R_{TP} &= \prod_{i=1}^{N_T} (1 + R_{iP}) \\ &= \prod_{i=1}^{N_T} \sum_{j=1}^{N_P} w_{ij}(1 + R_{ij}). \end{aligned} \tag{28}$$

As done in [1], it is reasonable to consider the case where the weights $w_{ij}$ are fixed for each period so that we can replace $w_{ij}$ with $w_j$. In this restricted case, the order of the above summation and product may be reversed giving

$$1 + R_{TP} = \sum_{j=1}^{N_T} w_j (1 + R_{Tj}), \tag{29}$$

where

$$1 + R_{Tj} := \prod_{i=1}^{N_T} (1 + R_{ij}) \tag{30}$$

is the total return for the $j$th sector across all $N_T$ periods. Once again we see that we may either first move across sectors to find the portfolio return $R_{iP}$ for the $i$th period and then move across periods to find the total return $R_{TP}$, or we may first move across the periods to find the total return $R_{Tj}$ for the $j$th sector across all $N_T$ periods and then move across all sectors to determine the total portfolio return $R_{TP}$. However, the difficulty lies in the fact that the process of moving across sectors is not geometric, i.e. it involves summation rather than products. Therefore, dropping the restriction of constant weights, the total relative geometric performance may be written in the form

$$\mathbf{P}_{TP}^{G} = \prod_{i=1}^{N_T} \frac{1 + R_{iP}}{1 + \overline{R}_{iP}} = \prod_{i=1}^{N_T} \frac{\sum_{j=1}^{N_P} w_{ij}(1 + R_{ij})}{\sum_{j=1}^{N_P} \overline{w}_{ij}(1 + \overline{R}_{ij})}. \tag{31}$$

Comparing this with Equation (24), we see that the desired relationship in order to have a fully geometric approach to performance attribution would be

$$\prod_{j=1}^{N_P} \mathbf{P}_{ij}^{G} = \frac{1 + R_{iP}}{1 + \overline{R}_{iP}} = \frac{\sum_{j=1}^{N_P} w_{ij}(1 + R_{ij})}{\sum_{j=1}^{N_P} \overline{w}_{ij}(1 + \overline{R}_{ij})}. \tag{32}$$

This requirement in order to have a fully geometric approach to performance attribution may be subdivided into individual requirements for the factorized issue and sector selections in a straightforward manner giving

$$\prod_{j=1}^{N_P} (1 + \mathbf{I}_{ij}^{G}) = \frac{1 + R_{iP}}{1 + \widetilde{R}_{iP}} = \frac{\sum_{j=1}^{N_P} w_{ij}(1 + R_{ij})}{\sum_{j=1}^{N_P} w_{ij}(1 + \widetilde{R}_{ij})} \tag{33}$$

and

$$\prod_{j=1}^{N_P} (1 + \mathbf{S}_{ij}^{G}) = \frac{1 + \widetilde{R}_{iP}}{1 + \overline{R}_{iP}} = \frac{\sum_{j=1}^{N_P} w_{ij}(1 + \widetilde{R}_{ij})}{\sum_{j=1}^{N_P} \overline{w}_{ij}(1 + \overline{R}_{ij})}. \tag{34}$$

The task of explicitly deriving an expression for the issue and sector selection was accomplished in [1] by incorporating an extra degree of freedom into corrective factors $\Gamma_{iP}^{GI}$ and $\Gamma_{iP}^{GS}$ so that the individual issue and sector selections take on the slightly modified forms

$$1 + \mathbf{I}_{ij}^{G} = \left( \frac{1 + w_{ij} R_{ij}}{1 + w_{ij} \overline{R}_{ij}} \right) \Gamma_{iP}^{GI} \tag{35}$$

and

$$1 + \mathbf{S}_{ij}^{G} = \left( \frac{1 + w_{ij} \overline{R}_{ij}}{1 + \overline{w}_{ij} \overline{R}_{ij}} \right) \left( \frac{1 + \overline{w}_{ij} \overline{R}_{iP}}{1 + w_{ij} \overline{R}_{iP}} \right) \Gamma_{iP}^{GS}. \tag{36}$$

It is then straightforward to determine the corrective factors by enforcing Equations (33) and (34) giving

$$\Gamma_{iP}^{GI} = \left[ \frac{1 + R_{iP}}{1 + \widetilde{R}_{iP}} \prod_{j=1}^{N_P} \left( \frac{1 + w_{ij} \overline{R}_{ij}}{1 + w_{ij} R_{ij}} \right) \right]^{\frac{1}{N_P}} \tag{37}$$

and

$$\Gamma_{iP}^{GS} = \left[ \frac{1 + \widetilde{R}_{iP}}{1 + \overline{R}_{iP}} \prod_{j=1}^{N_P} \left( \frac{1 + \overline{w}_{ij} \overline{R}_{ij}}{1 + w_{ij} \overline{R}_{ij}} \right) \left( \frac{1 + w_{ij} \overline{R}_{iP}}{1 + \overline{w}_{ij} \overline{R}_{iP}} \right) \right]^{\frac{1}{N_P}}. \tag{38}$$

It is now possible to write down an explicit expression for the localized relative geometric performance for the $j$th sector during the $i$th period given by

$$\mathbf{P}_{ij}^{G} = \left( \frac{1 + w_{ij} R_{ij}}{1 + \overline{w}_{ij} \overline{R}_{ij}} \right) \left( \frac{1 + \overline{w}_{ij} \overline{R}_{iP}}{1 + w_{ij} \overline{R}_{iP}} \right) \Gamma_{iP}^{G}, \tag{39}$$

where

$$\Gamma_{iP}^{G} = \Gamma_{iP}^{GI}\Gamma_{iP}^{GS} = \left[ \frac{1+R_{iP}}{1+\overline{R}_{iP}} \prod_{j=1}^{N_P} \left( \frac{1+\overline{w}_{ij}\overline{R}_{ij}}{1+w_{ij}R_{ij}} \right) \left( \frac{1+w_{ij}\overline{R}_{iP}}{1+\overline{w}_{ij}\overline{R}_{iP}} \right) \right]^{\frac{1}{N_P}}. \tag{40}$$

Equation (39) represents an explicit expression for the fully geometric relative performance for the $j$th sector during the $i$th period as suggested by [1]. It is then possible to perform both issue and sector selection analysis for either the entire portfolio at a given period or for an single sector over all periods. It is this versatility that makes the fully geometric approach to portfolio performance attribution so useful.

Although Menchero's method is geometrically linkable across sectors as well as across periods, the expressions for the relative performance are far from being intuitive. Furthermore, the geometric attribution suffers an anomaly that may introduce a slight bias as demonstrated in the following Section.

## 2 An Extension of the Fully Geometric Approach

When taking concepts from arithmetic attribution and converting them to geometric attribution, the approach taken by Menchero in [1] was to first expand all products. Then each term $X$ in the expansion is replaced with $(1 + X)$, addition is replaced with multiplication, and subtraction is replaced with division. For example, an algebraic expression

$$f^A = W + X - Y - Z \tag{41}$$

is translated into the geometric expression

$$f^G = \frac{(1+W)(1+X)}{(1+Y)(1+Z)}. \tag{42}$$

In particular, the arithmetic issue selection

$$\mathbf{I}_{ij}^{A} = w_{ij}(R_{ij} - \overline{R}_{ij}) \tag{43}$$

was translated into the proposed geometric issue selection

$$\mathbf{I}_{ij}^{G} = \frac{1+w_{ij}R_{ij}}{1+w_{ij}\overline{R}_{ij}}\Gamma_{iP}^{IG}, \tag{44}$$

while the arithmetic sector selection

$$\mathbf{S}_{ij}^{A} = (w_{ij} - \overline{w}_{ij})(\overline{R}_{ij} - \overline{R}_{iP}) \tag{45}$$

was translated into the proposed geometric sector selection

$$1 + \mathbf{S}_{ij}^{G} = \left( \frac{1+w_j\overline{R}_{ij}}{1+\overline{w}_{ij}\overline{R}_{ij}} \right) \left( \frac{1+\overline{w}_{ij}\overline{R}_{iP}}{1+w_j\overline{R}_{iP}} \right) \Gamma_{iP}^{SG}. \tag{46}$$

The translation of the arithmetic issue selection is perfectly natural. However, the translation of the arithmetic sector selection introduces a slight anomaly. To see how this comes about, recall that the terms involving the benchmark return $\overline{R}_{iP}$ drop out once you perform a sum over all sectors, i.e.

$$\sum_{j=1}^{N_P} \mathbf{S}_{ij}^{A} = \sum_{j=1}^{N_P} (w_{ij} - \overline{w}_{ij})(\overline{R}_{ij} - \overline{R}_{iP}) \\ = \sum_{j=1}^{N_P} (w_{ij} - \overline{w}_{ij})\overline{R}_{ij}. \tag{47}$$

On the other hand, the benchmark return $\overline{R}_{iP}$ does not drop out of the geometric sector selection when a product is taken over all sectors, i.e.

$$\prod_{j=1}^{N_P} (1 + \mathbf{S}_{ij}^{G}) = \prod_{j=1}^{N_P} \left( \frac{1 + w_j \overline{R}_{ij}}{1 + \overline{w}_{ij} \overline{R}_{ij}} \right) \left( \frac{1 + \overline{w}_{ij} \overline{R}_{iP}}{1 + w_j \overline{R}_{iP}} \right) \Gamma_{iP}^{GS} \\ \neq \left( \Gamma_{iP}^{GS} \right)^{N_P} \prod_{j=1}^{N_P} \frac{1 + w_j \overline{R}_{ij}}{1 + \overline{w}_{ij} \overline{R}_{ij}}. \tag{48}$$

Having the benchmark return $\overline{R}_{iP}$ drop out of the product is desirable in order to have the geometric sector selection more closely parallel the behavior of the arithmetic sector selection and thus avoid introducing bias.

With this in mind, introduce an alternative geometric sector selection

$$1 + \mathbf{S}_{ij}^{E} := \left( \frac{1 + w_j \overline{R}_{ij}}{1 + \overline{w}_{ij} \overline{R}_{ij}} \right) \exp\left[ -(w_{ij} - \overline{w}_{ij}) \overline{R}_{iP} \right] \Gamma_{iP}^{ES}, \tag{49}$$

so that

$$\prod_{j=1}^{N_P} (1 + \mathbf{S}_{ij}^{E}) = \prod_{j=1}^{N_P} \left( \frac{1 + w_j \overline{R}_{ij}}{1 + \overline{w}_{ij} \overline{R}_{ij}} \right) \exp\left[ -(w_{ij} - \overline{w}_{ij}) \overline{R}_{iP} \right] \Gamma_{iP}^{ES} \\ = \left( \Gamma_{iP}^{ES} \right)^{N_P} \prod_{j=1}^{N_P} \left( \frac{1 + w_j \overline{R}_{ij}}{1 + \overline{w}_{ij} \overline{R}_{ij}} \right). \tag{50}$$

Then, enforcing the relation

$$\prod_{j=1}^{N_P} (1 + \mathbf{S}_{ij}^{E}) = \frac{1 + \widetilde{R}_{iP}}{1 + \overline{R}_{iP}} \tag{51}$$

allows us to solve for the new correction factor

$$\Gamma_{iP}^{ES} := \left[ \frac{1 + \widetilde{R}_{iP}}{1 + \overline{R}_{iP}} \prod_{j=1}^{N_P} \left( \frac{1 + \overline{w}_{ij} \overline{R}_{ij}}{1 + w_j \overline{R}_{ij}} \right) \right]^{\frac{1}{N_P}}. \tag{52}$$

The choice of Equation (49) for the geometric sector selection provides for a coherent analysis of the portfolio managers selections of weights $w_{ij}$ that is free from the slight offset caused by the subtle inconsistency that ensues from translating the arithmetic sector selection directly to the geometric sector selection. Not only does the resulting geometric sector selection more accurately quantify the performance of the portfolio manager, it also provides for simpler expressions. The extended geometric relative performance for the $j$th sector during the $i$th period may now be written as

$$\begin{aligned} \mathbf{P}_{ij}^{GE} &= (1+\mathbf{I}_{ij}^{G})(1+\mathbf{S}_{ij}^{E}) \\ &= \left(\frac{1+w_{ij}R_{ij}}{1+\overline{w}_{ij}\overline{R}_{ij}}\right)\exp\left[-(w_{ij}-\overline{w}_{ij})\overline{R}_{iP}\right]\Gamma_{iP}^{GE}, \end{aligned} \tag{53}$$

where

$$\Gamma_{iP}^{GE} := \Gamma_{iP}^{GI}\Gamma_{iP}^{ES} = \left[\frac{1+R_{iP}}{1+\overline{R}_{iP}}\prod_{j=1}^{N_P}\left(\frac{1+\overline{w}_{ij}\overline{R}_{ij}}{1+w_jR_{ij}}\right)\right]^{\frac{1}{N_P}}. \tag{54}$$

The resulting geometric relative performance may be thought of as a slight extension of that presented in [1].

Although, this new approach is certainly cleaner than the expressions found in [1], there is still lacking a very clear intuitive picture as to the meaning of the performance measure being used. To address this important issue, the following Section presents a new fully geometric approach that has the advantage of being simple to understand as well as providing for transparent parallels to the arithmetic approach to performance attribution.

# 3 An Extension of the Exponential Approach

Motivated by the desirable qualities found in the extended geometric approach of the previous Section, consider another approach which more closely parallels the arithmetic approach and may be thought of as an extension of the exponential approach presented by Carino in [2].

Recall that in translating concepts from arithmetic attribution to geometric attribution, addition was translated to multiplication and subtraction was translated to division. Clearly, there is a natural way to achieve this same transformation via the exponential function. Therefore, let the geometric issue and sector selection be given by the expressions

$$1+\mathbf{I}_{ij}^{N} := \exp\left(\mathbf{I}_{ij}^{A}\right)\Gamma_{iP}^{NI} \tag{55}$$

and

$$1+\mathbf{S}_{ij}^{N} := \exp(\mathbf{S}_{ij}^{A})\Gamma_{iP}^{NS}. \tag{56}$$

The correction factors $\Gamma_{iP}^{NI}$ and $\Gamma_{iP}^{NS}$ may be determined by enforcing the relations

$$\prod_{j=1}^{N_P}(1+\mathbf{I}_{ij}^{N}) = \left(\Gamma_{iP}^{NI}\right)^{N_P}\exp\left(\mathbf{I}_{iP}^{A}\right) = \frac{1+R_{iP}}{1+\widetilde{R}_{iP}} \tag{57}$$

and

$$\prod_{j=1}^{N_P}(1+\mathbf{S}_{ij}^{N}) = \left(\Gamma_{iP}^{NS}\right)^{N_P}\exp\left(\mathbf{S}_{iP}^{A}\right) = \frac{1+\widetilde{R}_{iP}}{1+\overline{R}_{iP}} \tag{58}$$

resulting in

$$\Gamma^{NI}_{iP} = \left[\frac{1+R_{iP}}{1+\widetilde{R}_{iP}}\right]^{\frac{1}{N_P}} \exp\left(-\left\langle \mathbf{I}^A_{iP}\right\rangle\right), \tag{59}$$

where

$$\left\langle \mathbf{I}^A_{iP}\right\rangle = \frac{\mathbf{I}^A_{iP}}{N_P} = \frac{1}{N_P}\sum_{j=1}^{N_P} \mathbf{I}^A_{ij} \tag{60}$$

is the mean of the arithmetic issue selections and

$$\Gamma^{NS}_{iP} = \left[\frac{1+\widetilde{R}_{iP}}{1+\overline{R}_{iP}}\right]^{\frac{1}{N_P}} \exp\left(-\left\langle \mathbf{S}^A_{iP}\right\rangle\right). \tag{61}$$

where

$$\left\langle \mathbf{S}^A_{iP}\right\rangle = \frac{\mathbf{S}^A_{iP}}{N_P} = \frac{1}{N_P}\sum_{j=1}^{N_P} \mathbf{S}^A_{ij} \tag{62}$$

is the mean of the arithmetic sector selections. Therefore, the issue and sector selections may be written in the suggestive form

$$1+\mathbf{I}^N_{ij} = \left[\frac{1+R_{iP}}{1+\widetilde{R}_{iP}}\right]^{\frac{1}{N_P}} \exp\left(\mathbf{I}^A_{ij} - \left\langle \mathbf{I}^A_{iP}\right\rangle\right) \tag{63}$$

and

$$1+\mathbf{S}^N_{ij} = \left[\frac{1+\widetilde{R}_{iP}}{1+\overline{R}_{iP}}\right]^{\frac{1}{N_P}} \exp\left(\mathbf{S}^A_{ij} - \left\langle \mathbf{S}^A_{iP}\right\rangle\right). \tag{64}$$

The geometric relative performance of this modified exponential approach for the $j$th sector during the $i$th period may now be written simply as

$$\begin{aligned}\mathbf{P}^N_{ij} &= (1+\mathbf{I}^N_{ij})(1+\mathbf{S}^N_{ij}) \\ &= \left[\frac{1+R_{iP}}{1+\overline{R}_{iP}}\right]^{\frac{1}{N_P}} \exp\left(\mathbf{P}^A_{ij} - \left\langle \mathbf{P}^A_{iP}\right\rangle\right),\end{aligned} \tag{65}$$

where

$$\left\langle \mathbf{P}^A_{iP}\right\rangle = \frac{\mathbf{P}^A_{iP}}{N_P} = \frac{1}{N_P}\sum_{j} \mathbf{P}^A_{ij} \tag{66}$$

is the mean of the localized arithmetic attribution

$$\mathbf{P}^A_{ij} := \mathbf{I}^A_{ij} + \mathbf{S}^A_{ij} \tag{67}$$

across sectors.

The simplicity of the resulting expressions make this approach clearly superior to both Menchero's and Carino's approaches to geometric performance attribution. The expressions do not contain messy correction coefficients. The final expressions are all there is. Furthermore, the interpretion is absolutely obvious.

To see this, first of all note that

$$\lim_{N_P \to \infty} X^{\frac{1}{N_P}} = 1. \tag{68}$$

for any finite positive real number $X$. The closer $X$ is to one, the faster the limit converges. Also note that none of the terms

$$\frac{1+R_{iP}}{1+\widetilde{R}_{iP}}, \quad \frac{1+\widetilde{R}_{iP}}{1+\overline{R}_{iP}}, \quad \text{or} \quad \frac{1+R_{iP}}{1+\overline{R}_{iP}} \tag{69}$$

will be significantly differently from one for most situations. This means that

$$1 + \mathbf{I}^N_{ij} \approx \exp\left(\mathbf{I}^A_{ij} - \left\langle \mathbf{I}^A_{iP} \right\rangle\right), \tag{70}$$

$$1 + \mathbf{S}^N_{ij} \approx \exp\left(\mathbf{S}^A_{ij} - \left\langle \mathbf{S}^A_{iP} \right\rangle\right), \tag{71}$$

$$\mathbf{P}^N_{ij} \approx \exp\left(\mathbf{P}^A_{ij} - \left\langle \mathbf{P}^A_{iP} \right\rangle\right). \tag{72}$$

As the number of sectors $N_P$ in the portfolio increases, the approximation becomes better and better. In this way, the localized geometric issue selection, sector selection, and relative performance may all be interpretted as the exponential of the difference of the corresponding arithmetic parameter and the corresponding mean. This makes perfect sense. If a sector's relative performance is the same as the mean performance for that period, that sector gets assigned a performance measure of 1. If the the sector's relative performance is above the mean, it gets assigned a relative performance greater than one, etc. Exactly as it should be.

When the number of sectors across the portfolio is not large, then the corresponding factors in Equation (69) exactly compensate for the deviation from the exponential of the corresponding arithmetic parameters so that you maintain the desired geometrical linkage across sectors and periods for both issue and sector selection as well as the overall relative performance.

# 4 Conclusion

In this paper, a thorough review of Menchero's [1] fully geometric approach for performance attribution was given in Section 1. In Section 2, a slight improvement upon Menchero's approach was provided which may be viewed as a hybridization of the fully geometric approach and Carino's [2] exponential approach. In Section 3, a vast improvement over both Menchero's and Carino's approaches was introduced. The primary difference between the extended exponential approach of Section 3 and that of [2] is that here there are two corrective factors which allow you simultaneously analyze the geometric issue and sector selection performance in such a way that provides for a clear parallel between the arithmetic and geometric approaches.

# A A Note on Statistics and Possible Future Work

A further advantage of the extended geometric approach presented here is that it lends itself more naturally to a statistical analysis. Note that given a random variable $X$, then

$\exp(X - \langle X \rangle)$ is also a random variable. However,

$$\langle \exp(X - \langle X \rangle) \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle (X - \langle X \rangle)^n \rangle = 1 + \sum_{n=2}^{\infty} \frac{1}{n!} \langle (X - \langle X \rangle)^n \rangle \,, \tag{73}$$

which is clearly a sum of the higher moments. Since the extended exponential approach of Section 3 is of this form, it may be worth while persuing a statistical analysis of performance attribution. After all, none of the approaches to performance attribution presented here (Menchero's, Carino's, or the present approach) take into consideration the underlying risk. Should a portfolio manager achieve high marks for getting a high return on an extremely risky investment?

## References

[1] J. G. Menchero, "A fully geometric approach to performance attribution," *The Journal of Performance Management*, pp. 22–30, Winter 2000/2001.

[2] D. Carino, "Combining attribution effects over time," *Journal of Performance Measurement*, pp. 5–14, Summer 1999.