Noncommutative Geometry and Stochastic Calculus: Applications in Mathematical Finance

Eric A. Forgy

May 20, 2002

Abstract

The present report contains an introduction to some elementary concepts in noncommutative differential geometry. The material extends upon ideas first presented by Dimakis and Müller-Hoissen. In particular, stochastic calculus and the Ito formula are shown to arise naturally from introducing noncommutativity of functions (0-forms) and differentials (1-forms). The abstract construction allows for the straightforward generalization to lattice theories for the direct implementation of numerical models. As an elementary demonstration of the formalism, the standard Black-Scholes model for option pricing is reformulated.

Contents

1 Introduction

The present report introduces some of the elementary concepts of noncommutative geometry of commutative rings with an emphasis on its relation to stochastic calculus. For additional information the reader is encouraged to explore the wonderfully lucid introduction by Müller-Hoissen [1]. In Müller-Hoissen's review article, he mentions (almost in passing) that the Ito formula of stochastic calculus naturally arises from the introduction of noncommutativity of functions (0-forms) and differentials (1-forms) and refers to an earlier paper with Dimakis [2]. In the latter paper, the author's demonstrate in some detail the relation between noncommutative geometry and stochastic calculus. In particular, they derive the Ito formula from noncommutative geometry without any mention of Brownian motion whatsoever. This suggests a deep relation between stochastic processes and noncommutativity. That the two are deeply rooted in each other is not so surprising considering that each has played a significant role in quantum mechanics.

With an eye toward applications in mathematical finance, the present paper further explores the relationship between noncommutative geometry and stochastic calculus. As a simple application of the construction, the Black-Scholes model of option pricing is reformulated within the framework of noncommutative geometry. It should be noted that the formalism is in no way limited to the standard Black-Scholes model. The construction is quite general and can accommodate general models. In fact, the Black-Scholes model is a very special case.

Section 2 provides a quick overview of the basic definitions. Noncommutative differential calculus is introduced in its most abstract form. The important feature to notice here is that no assumptions about the underlying commutative ring are made. It need not be that of smooth scalar functions and can be any commutative ring. It is this generality that will allow for the adaptation to robust numerical methods on general cell complexes. Section 2.2 considers differential calculi that admit a coordinate basis. The differential calculi are seen to be characterized by commutation relations among 0-forms and 1-forms. Although the differential calculus may be characterized in terms of a particular coordinate system, Section 2.3 demonstrates that under very natural conditions, the differential calculus is independent of the coordinates chosen.

Section 3 considers the special case where 0-forms and 1-forms commute. Under these conditions, the differential calculus is referred to as an exterior differential calculus. It is shown that the usually calculus of differential forms is recovered, albeit over general commutative rings.

Section 4 introduces the stochastic differential calculus. The stochastic differential calculus is seen to be a special case of the general noncommutative differential calculus. Several novel concepts arise due to the general noncommutativity of 0-forms and 1-forms. In particular, the concepts of left and right martingales appear naturally. A 0-form is seen to be a right martingale if it satisfies the heat equation and is a left martingale if it satisfies the time-reversed heat equation.

Finally, Section 5 discusses the Black-Scholes model within the framework of noncommutative geometry. It is seen that care must be taken in proposing models for the value of certain assets. Enforcing consistency with the Ito formula singles out a preferred model for the spot price of a tradeable asset. Once the standard requirements of the Black-Scholes model are enforced, e.g. self-financing and no arbitrage, the usual Black-Scholes equation is derived for the value of an option as a function of the value of the underlying and time.

2 Noncommutative Differential Calculus

Let

$$\Omega := \bigoplus_{r=0}^{\infty} \Omega^r \tag{1}$$

denote a $\mathbb{Z}$-graded algebra. Recall that a $\mathbb{Z}$-graded algebra consists of a commutative[1] ring Ω^0 and a collection of Ω^0-modules Ω^r indexed by integers $r \in \mathbb{Z}$ together with an associative product

$$m : \Omega^p \times \Omega^q \to \Omega^{p+q} \tag{2}$$

and a unit element $\mathbf{1} \in \Omega^0$ satisfying

$$m(\mathbf{1}, \alpha) = m(\alpha, \mathbf{1}) = \alpha \tag{3}$$

for all $\alpha \in \Omega^r$. For simplicity, the product will be written simply as a juxtaposition, i.e.

$$\alpha\beta := m(\alpha, \beta) \tag{4}$$

for any $\alpha \in \Omega^p$ and $\beta \in \Omega^q$.

An element of Ω^p will be referred to as a p-form and the integer p is referred to as the degree of the form. Let $|\alpha|$ denote the degree of α, i.e. if $\alpha \in \Omega^p$, then $|\alpha| = p$. A derivation on a $\mathbb{Z}$-graded algebra Ω is a linear map $\mathrm{d} : \Omega^p \to \Omega^{p+1}$ satisfying the graded Leibniz rule

$$\mathrm{d}(\alpha\beta) = (\mathrm{d}\alpha)\beta + (-1)^{|\alpha|}\alpha(\mathrm{d}\beta) \tag{5}$$

and the nilpotency property

$$\mathrm{d}^2\alpha = 0 \tag{6}$$

for all $\alpha, \beta \in \Omega$. A $\mathbb{Z}$-graded algebra Ω together with a derivation d is referred to as a differential calculus (Ω, d) if

$$\Omega^{p+1} = \{f\mathrm{d}\alpha | f \in \Omega^0, \alpha \in \Omega^p\} \tag{7}$$

for $p \geq 0$.

2.1 Graded Lie Bracket

Given a $\mathbb{Z}$-graded algebra Ω, define the graded Lie bracket of forms via

$$[\alpha, \beta] := \alpha\beta - (-1)^{|\alpha||\beta|}\beta\alpha. \tag{8}$$

Then for a differential calculus (Ω, d), the graded Lie bracket satisfies the relations

$$\mathrm{d}[\alpha, \beta] = [\mathrm{d}\alpha, \beta] + (-1)^{|\alpha|}[\alpha, \mathrm{d}\beta]. \tag{9}$$

[1] Although it may be of interest to consider noncommutative rings, only commutative rings will be discussed in this report.

In particular, let f and g be 0-forms, then

$$\mathrm{d}[f,g] = [\mathrm{d}f,g] + [f,\mathrm{d}g] = 0 \tag{10}$$

so that

$$[\mathrm{d}f,g] = [\mathrm{d}g,f]. \tag{11}$$

The above identity will be used frequently in what follows.

2.2 Coordinate Bases

Consider an n-dimensional noncommutative differential calculus (Ω,d) with coordinate bases $\mathrm{d}x^\mu$ spanning Ω^1. The differential calculus may be characterized by the commutation relations[2]

$$[\mathrm{d}x^\mu, x^\nu] = \overleftarrow{C}^{\mu\nu}_{\lambda}\mathrm{d}x^\lambda, \tag{12}$$

where $\overleftarrow{C}^{\mu\nu}_{\lambda} \in \Omega^0$ are the left structure constants. Note that the structure constants are symmetric in the upper indices, i.e. $\overleftarrow{C}^{\mu\nu}_{\lambda} = \overleftarrow{C}^{\nu\mu}_{\lambda}$, which follows from the fact the $[\mathrm{d}x^\mu, x^\nu] = [\mathrm{d}x^\nu, x^\mu]$. A general 1-form α may then be written as

$$\alpha = \overleftarrow{\alpha}_\mu \mathrm{d}x^\mu, \tag{13}$$

where $\overleftarrow{\alpha}_\mu \in \Omega^0$ are the left components of α. In particular, the 1-form $\mathrm{d}f$ may be written as

$$\mathrm{d}f = (\overleftarrow{\partial}_\mu f)\mathrm{d}x^\mu, \tag{14}$$

where $\overleftarrow{\partial}_\mu : \Omega^0 \to \Omega^0$ is a linear map defined in such a way that Equation (14) is satisfied.

Definition 1. *A partial derivative is any map $D_\mu : \Omega^0 \to \Omega^0$ satisfying the product rule*

$$D_\mu(fg) = (D_\mu f)g + f(D_\mu g) \tag{15}$$

for all $f, g \in \Omega^0$ and the additional condition

$$D_\mu x^\nu = \delta^\nu_\mu, \tag{16}$$

where δ^ν_μ is the Kronecker delta.

Despite the suggestive notation, it should be pointed out that $\overleftarrow{\partial}_\mu$ is not in general a partial derivative with respect to x^μ. To see this, simply compute

$$\begin{aligned}
\mathrm{d}(fg) &= (\mathrm{d}f)g + f(\mathrm{d}g) \\
&= (\overleftarrow{\partial}_\lambda f)\mathrm{d}x^\lambda g + f(\overleftarrow{\partial}_\lambda g)\mathrm{d}x^\lambda \\
&= (\overleftarrow{\partial}_\lambda f)(g\mathrm{d}x^\lambda + [\mathrm{d}x^\lambda, g]) + f(\overleftarrow{\partial}_\lambda g)\mathrm{d}x^\lambda \\
&= \left[(\overleftarrow{\partial}_\lambda f)g + f(\overleftarrow{\partial}_\lambda g)\right]\mathrm{d}x^\lambda + (\overleftarrow{\partial}_\lambda f)(\overleftarrow{\partial}_\nu g)[\mathrm{d}x^\nu, x^\lambda] \\
&= \left[(\overleftarrow{\partial}_\lambda f)g + f(\overleftarrow{\partial}_\lambda g)\right]\mathrm{d}x^\lambda + (\overleftarrow{\partial}_\lambda f)(\overleftarrow{\partial}_\nu g)\overleftarrow{C}^{\lambda\nu}_{\mu}\mathrm{d}x^\mu \\
&= \left[(\overleftarrow{\partial}_\lambda f)g + f(\overleftarrow{\partial}_\lambda g) + (\overleftarrow{\partial}_\mu f)(\overleftarrow{\partial}_\nu g)\overleftarrow{C}^{\mu\nu}_{\lambda}\right]\mathrm{d}x^\lambda \\
&= \overleftarrow{\partial}_\lambda(fg)\mathrm{d}x^\lambda
\end{aligned} \tag{17}$$

[2]Einstein summation convention is implied so that when a matching subscript and superscript appear in the same term, a summation is implied, e.g. $\alpha_\mu \mathrm{d}x^\mu := \sum_{\mu=1}^n \alpha_\mu \mathrm{d}x^\mu$.

so that

$$\overleftarrow{\partial}_\lambda(fg) = (\overleftarrow{\partial}_\lambda f)g + f(\overleftarrow{\partial}_\lambda g) + (\overleftarrow{\partial}_\mu f)(\overleftarrow{\partial}_\nu g)\overleftarrow{C}^{\mu\nu}_\lambda. \tag{18}$$

Equation (18) clearly does not satisfy the product rule that would be required of a partial derivative when $\overleftarrow{C}^{\mu\nu}_\lambda \neq 0$.

The commutation relation for general 0-forms and 1-forms may then be deduced by considering

$$\begin{aligned}[\mathrm{d}x^\mu, f] &= [\mathrm{d}f, x^\mu] \\ &= (\overleftarrow{\partial}_\nu f)[\mathrm{d}x^\nu, x^\mu] \\ &= (\overleftarrow{\partial}_\nu f)\overleftarrow{C}^{\nu\mu}_\lambda \mathrm{d}x^\lambda \end{aligned} \tag{19}$$

so that

$$[\alpha, f] = \overleftarrow{\alpha}_\mu(\overleftarrow{\partial}_\nu f)\overleftarrow{C}^{\nu\mu}_\lambda \mathrm{d}x^\lambda \tag{20}$$

for any $\alpha \in \Omega^1$ and $f \in \Omega^0$.

2.3 Coordinate Independence

A natural question at this point would be to ask, "What happens under a change of coordinates?" In this Section, it will be shown that the differential calculus is invariant under general diffeomorphisms under very natural conditions.

Recall that an n-dimensional general noncommutative calculus (Ω, d) may be characterized by the commutation relations

$$[\mathrm{d}x^\mu, x^\nu] = \overleftarrow{C}^{\mu\nu}_\lambda \mathrm{d}x^\lambda. \tag{21}$$

Now consider an invertible change of coordinate bases defined via

$$\mathrm{d}x^\mu = \left(\overleftarrow{\partial}'_\nu x^\mu\right)\mathrm{d}x'^\nu, \tag{22}$$

where the transformation may be inverted resulting in[3]

$$\mathrm{d}x'^\nu = \left(\overleftarrow{\partial}'_\nu x^\mu\right)^{-1}\mathrm{d}x^\mu = \left(\overleftarrow{\partial}_\mu x'^\nu\right)\mathrm{d}x^\mu \tag{23}$$

Then the left side of the commutation relations in the new coordinates become

$$\begin{aligned}[\mathrm{d}x^\mu, x^\nu] &= \left(\overleftarrow{\partial}'_\kappa x^\mu\right)[\mathrm{d}x'^\kappa, x^\nu] \\ &= \left(\overleftarrow{\partial}'_\kappa x^\mu\right)\left(\overleftarrow{\partial}'_\lambda x^\nu\right)[\mathrm{d}x'^\lambda, x'^\kappa] \\ &= \left(\overleftarrow{\partial}'_\kappa x^\mu\right)\left(\overleftarrow{\partial}'_\lambda x^\nu\right)\overleftarrow{C}'^{\kappa\lambda}_\rho \mathrm{d}x'^\rho, \end{aligned} \tag{24}$$

while the right side becomes

$$\overleftarrow{C}^{\mu\nu}_\sigma \mathrm{d}x^\sigma = \left(\overleftarrow{\partial}'_\rho x^\sigma\right)\overleftarrow{C}^{\mu\nu}_\sigma \mathrm{d}x'^\rho. \tag{25}$$

[3] $\left(\overleftarrow{\partial}'_\nu x^\mu\right)^{-1}$ is the (μ,ν)-th entry of the inverse of the Jacobian matrix $\left(\overleftarrow{\partial}'_\nu x^\mu\right)$.

Comparing both sides results in

$$\overleftarrow{C}'^{\kappa\lambda}_{\rho} = \left(\overleftarrow{\partial}'_{\kappa} x^{\mu}\right)^{-1} \left(\overleftarrow{\partial}'_{\lambda} x^{\nu}\right)^{-1} \left(\overleftarrow{\partial}'_{\rho} x^{\sigma}\right) \overleftarrow{C}^{\mu\nu}_{\sigma}$$
$$= \left(\overleftarrow{\partial}_{\mu} x'^{\kappa}\right) \left(\overleftarrow{\partial}_{\nu} x'^{\lambda}\right) \left(\overleftarrow{\partial}'_{\rho} x^{\sigma}\right) \overleftarrow{C}^{\mu\nu}_{\sigma}. \quad (26)$$

Therefore, the differential calculus is independent of the coordinates chosen as long as the structure constants transform as the components of a mixed tensor of contravariant rank two and covariant rank one, i.e. a $(2, 1)$-tensor. In other words, the differential calculus is independent of coordinates chosen if the structure constants define a $(2, 1)$-tensor

$$\overleftarrow{C} := \overleftarrow{C}^{\mu\nu}_{\lambda} \mathrm{d}x^{\lambda} \otimes e_{\mu} \otimes e_{\nu}, \quad (27)$$

where e_{μ} are coordinate bases of the dual Ω^0-module $\left(\Omega^1\right)^*$ defined by

$$\langle \mathrm{d}x^{\mu}, e_{\nu} \rangle := \delta^{\mu}_{\nu} \quad (28)$$

and $\langle \quad , \quad \rangle : \Omega^1 \times \left(\Omega^1\right)^* \to \Omega^0$ is the evaluation map. The tensor $\overleftarrow{C}$ will be referred to as the structure tensor. Clearly, the structure tensor is a geometric object with inherent meaning independent of any chosen coordinate basis.

3 Exterior Differential Calculus

Before moving on to the more interesting examples, consider the simplest case of a differential calculus (Ω, d) where the structure tensor $\overleftarrow{C}$ is identically zero, i.e. all 0-forms and 1-forms commute. Such a differential calculus will be referred to as an exterior differential calculus.

Let (Ω, d) be an n-dimensional exterior differential calculus characterized by the commutation relation

$$[\alpha, f] = 0 \quad (29)$$

for all $\alpha \in \Omega^1$ and $f \in \Omega^0$. Then for any $x \in \Omega^0$, it follows that

$$\begin{aligned} \mathrm{d}(x^n) &= (\mathrm{d}x)x^{n-1} + x(\mathrm{d}(x^{n-1})) \\ &= (\mathrm{d}x)x^{n-1} + x(\mathrm{d}x)x^{n-2} + x^2(\mathrm{d}(x^{n-2})) \\ &= \sum_{i=0}^{n-1} x^i(\mathrm{d}x)x^{n-i-1} \\ &= \sum_{i=0}^{n-1} x^{n-1}\mathrm{d}x \\ &= nx^{n-1}\mathrm{d}x. \end{aligned} \quad (30)$$

This is precisely the expression from elementary analysis for the differential of a smooth scalar function. The difference here is that Ω^0 can be any commutative ring, not necessary that of smooth scalar functions.

Due to the fact that the structure constants vanish for a commutative differential calculus, the linear map ∂_μ (dropping the left arrow) satisfies the product rule

$$\partial_\mu(fg) = (\partial_\mu f)g + f(\partial_\mu g). \tag{31}$$

Furthermore,

$$\mathrm{d}x^\mu = (\partial_\nu x^\mu)\mathrm{d}x^\nu \tag{32}$$

implies that

$$\partial_\nu x^\mu = \delta^\mu_\nu. \tag{33}$$

Therefore, in the case of an exterior differential calculus, it is appropriate to think of ∂_μ as a partial derivative with respect to x^μ.

A general 1-form α may then be expressed via a coordinate basis as

$$\alpha = \alpha_\mu \mathrm{d}x^\mu, \tag{34}$$

where no distinction is required between left and right components. Since all 0-forms commute with all 1-forms in a commutative differential calculus, it then follows that

$$\begin{aligned} \mathrm{d}[x^\mu, \mathrm{d}x^\nu] &= [\mathrm{d}x^\mu, \mathrm{d}x^\nu] \\ &= \mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}x^\nu \mathrm{d}x^\mu \\ &= 0 \end{aligned} \tag{35}$$

so that

$$\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}x^\nu \mathrm{d}x^\mu. \tag{36}$$

Therefore, the product of 1-forms in an exterior differential calculus is anticommutative, i.e. $\alpha\beta = -\beta\alpha$ for all $\alpha, \beta \in \Omega^1$. It can further be shown that in general

$$\alpha\beta = (-1)^{|\alpha||\beta|}\beta\alpha. \tag{37}$$

Thus we recover both the algebraic and differential properties of the continuum theory of differential forms without ever specifying exactly what the commutative ring Ω^0 was. The only requirement was that all 0-forms commute with 1-forms in a differential calculus (Ω, d).

4 Stochastic Differential Calculus

Consider an $n+1$ dimensional differential calculus (Ω, d) with n spatial coordinate bases labeled $\mathrm{d}x^\mu$ and an additional temporal coordinate basis $\mathrm{d}t$ satisfying the commutative relations[4]

$$[\mathrm{d}x^\mu, t] = [\mathrm{d}t, t] = 0 \tag{38}$$

and

$$[\mathrm{d}x^\mu, x^\nu] = \mathrm{g}^{\mu\nu}\mathrm{d}t. \tag{39}$$

[4]Throughout this Section, $\mathrm{d}x^\mu \neq \mathrm{d}t$ for any μ.

In other words, the temporal coordinate basis $\mathrm{d}t$ is purely commutative, but noncommutativity has been introduced into the spatial coordinate bases $\mathrm{d}x^\mu$. Note that $[\mathrm{d}t, x^\mu] = [\mathrm{d}x^\mu, t] = 0$ so that $\mathrm{d}x^\mu$ commutes with t, but not in general with x^ν. In terms of the structure constants, this translates into

$$\overleftarrow{C}^{\mu\nu}_{\lambda} = \begin{cases} \mathrm{g}^{\mu\nu} & \text{for } \lambda = t \text{ and } \mu,\nu \neq t \\ 0 & \text{otherwise.} \end{cases} \tag{40}$$

In this case, the structure tensor takes on a suggestive form

$$\overleftarrow{C} = \mathrm{d}t \otimes \mathrm{g}, \tag{41}$$

where

$$\mathrm{g} = \mathrm{g}^{\mu\nu} e_\mu \otimes e_\nu \tag{42}$$

is a symmetric $(2,0)$-tensor.

A general 1-form α may then be written as

$$\alpha = \overleftarrow{\alpha}_\mu \mathrm{d}x^\mu + \overleftarrow{\alpha}_t \mathrm{d}t \tag{43}$$

with left components $\overleftarrow{\alpha}_x, \overleftarrow{\alpha}_t \in \Omega^0$. Alternatively, the 1-form α may be expressed in terms of the right components via

$$\alpha = \mathrm{d}x^\mu \overrightarrow{\alpha}_\mu + \mathrm{d}t \overrightarrow{\alpha}_t. \tag{44}$$

The distinction between left and right components is important here due to the noncommutativity of 0-forms and 1-forms. The 1-form $\mathrm{d}f$ may also be written as

$$\mathrm{d}f = (\overleftarrow{\partial}_\mu f)\mathrm{d}x^\mu + (\overleftarrow{\partial}_t f)\mathrm{d}t. \tag{45}$$

With the structure constants (40), the linear maps $\overleftarrow{\partial}_\mu : \Omega^0 \to \Omega^0$ satisfy the product rule as well as

$$\overleftarrow{\partial}_\mu x^\nu = \delta^\nu_\mu. \tag{46}$$

Hence, $\overleftarrow{\partial}_\mu$ may be thought of as a partial derivative with respect to x^μ.

On the other hand, the linear map $\overleftarrow{\partial}_t : \Omega^0 \to \Omega^0$ satisfies

$$\overleftarrow{\partial}_t(fg) = (\overleftarrow{\partial}_t f)g + f(\overleftarrow{\partial}_t g) + g^{\mu\nu}(\overleftarrow{\partial}_\mu f)(\overleftarrow{\partial}_\nu g). \tag{47}$$

Therefore, $\overleftarrow{\partial}_t$ may not be thought of as a partial derivative. However, define

$$\overleftarrow{\partial}_t := \partial_t + \frac{1}{2}\mathrm{g}^{\mu\nu}\overleftarrow{\partial}_\mu \overleftarrow{\partial}_\nu, \tag{48}$$

where $\partial_t : \Omega^0 \to \Omega^0$ satisfies the properties of a partial derivative with respect to t. Then

$$\begin{aligned} \overleftarrow{\partial}_t(fg) &= \partial_t(fg) + \frac{1}{2}\mathrm{g}^{\mu\nu}\overleftarrow{\partial}_\mu \overleftarrow{\partial}_\nu(fg) \\ &= (\partial_t f)g + f(\partial g) + \frac{1}{2}\mathrm{g}^{\mu\nu}\overleftarrow{\partial}_\mu \left[(\overleftarrow{\partial}_\nu f)g + f(\overleftarrow{\partial}_\nu g)\right] \\ &= \left[\partial_t f + \frac{1}{2}\mathrm{g}^{\mu\nu}\overleftarrow{\partial}_\mu \overleftarrow{\partial}_\nu f\right] g + f\left[\partial_t g + \frac{1}{2}\mathrm{g}^{\mu\nu}\overleftarrow{\partial}_\mu \overleftarrow{\partial}_\nu g\right] + \mathrm{g}^{\mu\nu}(\overleftarrow{\partial}_\mu f)(\overleftarrow{\partial}_\nu g) \\ &= (\overleftarrow{\partial}_t f)g + f(\overleftarrow{\partial}_t g) + \mathrm{g}^{\mu\nu}(\overleftarrow{\partial}_\mu f)(\overleftarrow{\partial}_\nu g) \end{aligned} \tag{49}$$

as desired. Finally, $\mathrm{d}f$ may be written as

$$\begin{aligned}
\mathrm{d}f &= (\overleftarrow{\partial}_\mu f)\mathrm{d}x^\mu + (\overleftarrow{\partial}_t f)\mathrm{d}t \\
&= (\overleftarrow{\partial}_\mu f)\mathrm{d}x^\mu + (\partial_t f + \frac{1}{2}\mathrm{g}^{\mu\nu}\overleftarrow{\partial}_\mu\overleftarrow{\partial}_\nu f)\mathrm{d}t. \qquad (50)
\end{aligned}$$

Equation (50) is referred to as the Ito formula in stochastic calculus. However, this expression was arrived at without any mention of stochastic processes at all. This result follows directly from the introduction of noncommutativity of 0-forms and 1-forms. Moreover, the ring Ω^0 has still not been specified. It could be any commutative ring.

The Ito formula of Equation (50) has been written in terms of the left components. Expressing the same result via right components may be done simply using the commutation relations resulting in

$$\begin{aligned}
\mathrm{d}f &= (\overleftarrow{\partial}_\mu f)\mathrm{d}x^\mu + (\overleftarrow{\partial}_t f)\mathrm{d}t \\
&= (\overleftarrow{\partial}_\mu f)\mathrm{d}x^\mu + (\partial_t f + \frac{1}{2}\mathrm{g}^{\mu\nu}\overleftarrow{\partial}_\mu\overleftarrow{\partial}_\nu f)\mathrm{d}t \\
&= \Big(\mathrm{d}x^\mu(\overleftarrow{\partial}_\mu f) - [\mathrm{d}x^\mu, (\overleftarrow{\partial}_\mu f)]\Big) + (\partial_t f + \frac{1}{2}\mathrm{g}^{\mu\nu}\overleftarrow{\partial}_\mu\overleftarrow{\partial}_\nu f)\mathrm{d}t \\
&= \mathrm{d}x^\mu(\overleftarrow{\partial}_\mu f) + \mathrm{d}t(\partial_t f - \frac{1}{2}\mathrm{g}^{\mu\nu}\overleftarrow{\partial}_\mu\overleftarrow{\partial}_\nu f) \\
&= \mathrm{d}x^\mu(\overrightarrow{\partial}_\mu f) + \mathrm{d}t(\overrightarrow{\partial}_t f) \qquad (51)
\end{aligned}$$

so that

$$\overrightarrow{\partial}_\mu = \overleftarrow{\partial}_\mu \qquad (52)$$

and

$$\overrightarrow{\partial}_t = \partial_t - \frac{1}{2}\mathrm{g}^{\mu\nu}\overrightarrow{\partial}_\mu\overrightarrow{\partial}_\nu. \qquad (53)$$

Due to the fact that $\overrightarrow{\partial}_\mu = \overleftarrow{\partial}_\mu$, there is no need to distinguish the two when discussing stochastic calculus. Therefore, let $\partial_\mu := \overrightarrow{\partial}_\mu = \overleftarrow{\partial}_\mu$. The Ito formula may now be written more cleanly in either the left component form

$$\mathrm{d}f = (\partial_\mu f)\mathrm{d}x^\mu + (\partial_t f + \frac{1}{2}\mathrm{g}^{\mu\nu}\partial_\mu\partial_\nu f)\mathrm{d}t \qquad (54)$$

or the right component form

$$\mathrm{d}f = \mathrm{d}x^\mu(\partial_\mu f) + \mathrm{d}t(\partial_t f - \frac{1}{2}\mathrm{g}^{\mu\nu}\partial_\mu\partial_\nu f). \qquad (55)$$

The fact that the left and right component expressions of Ito's formula are not equivalent is a direct consequence of noncommutativity of 0-forms and 1-forms. This is in stark contrast to the usual stochastic calculus where 0-forms and 1-forms do commute, but the price paid is the loss of the Leibniz rule as well as the nilpotency of d. Thus, when studying stochastic differentials equations, there is a choice to be made. You may either have 0-forms and 1-forms commute, but lose the Leibniz rule and the nilpotency of d as is usually done, or you may modify the product so that it is noncommutative while in the process retaining the Leibniz rule as well as the nilpotency of d. The latter is of consequence for cohomology as well as the related conservation laws.

Note also that the vanishing of the right temporal component in Ito's formula implies that f satisfies the heat equation, while vanishing of the left temporal component implies that f satisfies the time-reversed heat equation. This leads to the novel concepts of left and right martingales.

Definition 2. *A 0-form f whose right (left) temporal component of df vanishes when expressed in the coordinate bases (dx^μ, dt) is said to be a right (left) martingale with respect to (dx^μ, dt).*

As a consequence, a 0-form that satisfies the heat equation is a right martingale and a 0-form that satisfies the time-reversed heat equation is a left martingale in the stochastic differential calculus.

Before moving on to applications in the next Section, note that the Leibniz rule now becomes

$$\begin{aligned}
\mathrm{d}(fg) &= (\mathrm{d}f)g + f\mathrm{d}g \\
&= (g\mathrm{d}f + [\mathrm{d}f, g]) + f\mathrm{d}g \\
&= f\mathrm{d}g + g\mathrm{d}f + (\partial_\mu f)\,[\mathrm{d}x^\mu, g] \\
&= f\mathrm{d}g + g\mathrm{d}f + (\partial_\mu f)\,[\mathrm{d}g, x^\mu] \\
&= f\mathrm{d}g + g\mathrm{d}f + (\partial_\mu f)\,(\partial_\nu g)\,[\mathrm{d}x^\nu, x^\mu] \\
&= f\mathrm{d}g + g\mathrm{d}f + \mathrm{g}^{\mu\nu}\,(\partial_\mu f)\,(\partial_\nu g)\,\mathrm{d}t.
\end{aligned} \tag{56}$$

Again, this is a standard result in stochastic calculus, but here the difference from the standard product rule is seen to be due purely to noncommutativity and not due to randomness in any way.

5 The Black-Scholes Model

For the analysis of the Black-Scholes model with one commuting temporal coordinate basis and a second noncommuting coordinate basis dx, the commutation relations are taken to be

$$[\mathrm{d}t, f] = 0 \quad \text{and} \quad [\mathrm{d}x, x] = \mathrm{d}t \tag{57}$$

for all $f \in \Omega^0$. The first step in constructing the Black-Scholes model is to specify governing equation for the spot price S of a tradeable asset. The spot price S is typically modeled via

$$\mathrm{d}S = S\,(\sigma\mathrm{d}x + \mu\mathrm{d}t) \tag{58}$$

for constant positive real numbers $\sigma, \mu \in \mathbb{R}$. However, now that 0-forms and 1-forms do not commute, there is an apparent ambiguity. The spot price S could just as well have been assumed to be governed by the right component form

$$\mathrm{d}S = (\mathrm{d}x\sigma + \mathrm{d}t\mu)\,S. \tag{59}$$

In fact, a weighted average of the two may be taken, i.e.

$$\mathrm{d}S = \sigma\left[\kappa\left(S\mathrm{d}x\right) + (1-\kappa)\left(\mathrm{d}xS\right)\right] + \mu S\mathrm{d}t, \tag{60}$$

where $\kappa \in \mathbb{R}$ is simply some real weighting coefficient. Expressing the above in terms of left components results in

$$\begin{aligned}\mathrm{d}S &= \sigma\left[\kappa\left(S\mathrm{d}x\right) + (1-\kappa)\left(\mathrm{d}xS\right)\right] + \mu S\mathrm{d}t \\ &= \kappa\sigma S\mathrm{d}x + (1-\kappa)\left(S\mathrm{d}x + [\mathrm{d}x, S]\right) + \mu S\mathrm{d}t \\ &= \sigma S\mathrm{d}x + \left[\mu S + (1-\kappa)\sigma\left(\partial_x S\right)\right]\mathrm{d}t.\end{aligned} \tag{61}$$

Comparing this with the Ito formula

$$\mathrm{d}S = \left(\partial_x S\right)\mathrm{d}x + (\partial_t S + \frac{1}{2}\partial_x^2 S)\mathrm{d}t \tag{62}$$

demonstrates that the appropriate model is given by $\kappa = \frac{1}{2}$ with

$$\partial_x S = \sigma S \tag{63}$$

and

$$\partial_t S = \mu S. \tag{64}$$

Therefore, in order to have consistency with the Ito formula, the spot price S of the tradeable asset may be expressed in any one of three equivalent forms

$$\mathrm{d}S = \frac{\sigma}{2}\left(S\mathrm{d}x + \mathrm{d}xS\right) + \mu S\mathrm{d}t, \tag{65}$$

$$\mathrm{d}S = \sigma S\mathrm{d}x + \left(\mu + \frac{\sigma^2}{2}\right)S\mathrm{d}t, \tag{66}$$

or

$$\mathrm{d}S = \mathrm{d}xS\sigma + \mathrm{d}tS\left(\mu - \frac{\sigma^2}{2}\right). \tag{67}$$

The next ingredient in the Black-Scholes model is a risk-free bond whose value B is simply governed by

$$\mathrm{d}B = rB\mathrm{d}t. \tag{68}$$

There is no ambiguity in this definition.

The final ingredient in the Black-Scholes model is an option whose underlying is the original tradeable asset and whose value is denoted by V. The option value V is a function of the tradeable asset value S and time t. Therefore, it is desirable to express the differential $\mathrm{d}V$ in terms of $\mathrm{d}S$ and $\mathrm{d}t$. This may be done by first noting that

$$\partial_x V = (\partial_S V)(\sigma S) \tag{69}$$

and

$$\partial_x^2 V = (\partial_S^2 V)(\sigma S)^2 + (\partial_S V)(\sigma^2 S) \tag{70}$$

so that the Ito formula may be written in the alternative form

$$\begin{aligned}
\mathrm{d}V &= (\partial_x V)\mathrm{d}x + (\partial_t V + \frac{1}{2}\partial_x^2 V)\mathrm{d}t \\
&= (\partial_x V)\left[\frac{1}{\sigma S}\left(\mathrm{d}S - (\mu + \frac{\sigma^2}{2})S\mathrm{d}t\right)\right] + (\partial_t V + \frac{1}{2}\partial_x^2 V)\mathrm{d}t \\
&= (\partial_S V)\mathrm{d}S + \left(\partial_t V + \frac{\sigma^2 S^2}{2}\partial_S^2 V + \frac{\sigma^2}{2}S\partial_S V - (\mu + \frac{\sigma^2}{2})S\partial_S V\right)\mathrm{d}t \\
&= (\partial_S V)\mathrm{d}S + \left(\partial_t V + \frac{\sigma^2 S^2}{2}\partial_S^2 V - \mu S\partial_S V\right)\mathrm{d}t. \qquad (71)
\end{aligned}$$

Now, consider the term $\partial_t V$ above. This term was written down for the case when V is a function of x and t. Thus, $\partial_t V$ measures the change in V as t is varied while x is held fixed. However, if we express V in terms of S and t, then we would like $\partial_t V$ to measure the change in V as t is varied while S is held fixed. Therefore,

$$\begin{aligned}
\partial_t V(x,t) &= \partial_t V(S,t) + (\partial_S V(S,t))(\partial_t S(x,t)) \\
&= \partial_t V(S,t) + (\partial_S V(S,t))(\mu S(x,t)). \qquad (72)
\end{aligned}$$

Substituting this into Equation (71) results in the familiar form of Ito's formula

$$\mathrm{d}V = (\partial_S V)\mathrm{d}S + (\partial_t V + \frac{\sigma^2 S^2}{2}\partial_S^2 V)\mathrm{d}t. \qquad (73)$$

The total value Π of a portfolio consisting of α units of options, Δ units of tradeable assets, and β units of risk-free bonds is then given by

$$\Pi = \alpha V + \Delta S + \beta B. \qquad (74)$$

The differential $\mathrm{d}\Pi$ may be decomposed into

$$\mathrm{d}\Pi = \mathrm{d}M + \mathrm{d}T \qquad (75)$$

where

$$\mathrm{d}M = \alpha \mathrm{d}V + \Delta \mathrm{d}S + \beta \mathrm{d}B \qquad (76)$$

measures changes in the value of the portfolio due to changes in the market and

$$\mathrm{d}T = (\mathrm{d}\alpha)V + (\mathrm{d}\Delta)S + (\mathrm{d}\beta)B \qquad (77)$$

measures changes in the value of the portfolio due solely to the additional purchasing or selling of securities independent of the market. The Black-Scholes model assumes that changes in the value of the portfolio are due only to changes in the market so that $\mathrm{d}T = 0$, i.e. the portfolio is self financing. Therefore, the Black-Scholes model assumes

$$\mathrm{d}\Pi = \alpha \mathrm{d}V + \Delta \mathrm{d}S + \beta \mathrm{d}B. \qquad (78)$$

At this point, the derivation of the Black-Scholes equations follow directly from standard arguments [3]. The details are included below merely for completeness. The first step is to apply Ito's formula Equation (73) to $\mathrm{d}\Pi$ resulting in

$$\begin{aligned} \mathrm{d}\Pi &= \alpha \left[(\partial_S V)\mathrm{d}S + (\partial_t V + \frac{\sigma^2 S^2}{2}\partial_S^2 V)\mathrm{d}t \right] + \Delta \mathrm{d}S + \beta r B \mathrm{d}t \\ &= [\alpha(\partial_S V) + \Delta]\,\mathrm{d}S + \left[\alpha\partial_t V + \alpha\frac{\sigma^2 S^2}{2}\partial_S^2 V + \beta r B \right] \mathrm{d}t. \end{aligned} \tag{79}$$

The next step is the enforcement of the no arbitrage requirement, which essentially amounts to requiring that the portfolio value follow the risk-free rate via

$$\begin{aligned} \mathrm{d}\Pi &= r\Pi\mathrm{d}t \\ &= r(\alpha V + \Delta S + \beta B)\mathrm{d}t. \end{aligned} \tag{80}$$

First of all, note that there is no $\mathrm{d}S$ term above. Therefore, setting the two expressions for $\mathrm{d}\Pi$ equal to one another forces the condition

$$\Delta = -\alpha\partial_S V. \tag{81}$$

This condition provides for a hedging strategy so that the portfolio value follows the risk-free rate, i.e. it prescribes a method for determining the number of units of the underlying tradeable asset to hold as a function of how the option value changes with respect to the spot price of the underlying.

Finally, setting the two time components equal to one another results in the final form of the Black-Scholes equation for the value of the option as a function of S and t

$$\partial_t V + \frac{\sigma^2 S^2}{2}\partial_S^2 V + rS\partial_S V - rV = 0. \tag{82}$$

The value V of the option may then be determined by solving the above partial differential equation given the appropriate boundary conditions for the specific type of option under consideration.

6 Conclusions

In the present report, a general noncommutative differential calculus has been presented and both the exterior differential calculus and the stochastic differential calculus were shown to be special cases. Unlike the standard presentation of stochastic differential calculus, within the framework of noncommutative differential calculus, the derivation $\mathrm{d} : \Omega^p \to \Omega^{p+1}$ satisfies both the graded Leibniz rule as well as the nilpotency property $\mathrm{d}^2 = 0$. This is of great interest from the point of view of cohomology.

Although the subject is presently of little concern among practitioners in computational physics and/or finance, it is becoming increasingly recognized that in order to develop robust numerical algorithms that have the ability to efficiently solve large scale problems that the computational model mimic as closely as possible the underlying topological properties of the corresponding continuum theory. These topological properties are of fundamental importance when considering conservation laws. If there exists a conserved quantity in the continuum theory, e.g. charge, energy, etc., it would obviously be desirable to have the numerical model mimic this property exactly and not as an approximation. That this is possible with certain numerical models is a direct consequence of paying close attention to the underlying topological properties.

With this in mind, noncommutative differential geometry has a naturally extension to abstract cell complexes as presented in [4]. This promises to provide for a robust mathematical framework for constructing numerical models of stochastic differential equations in any number of dimensions. This, of course, would be of great interest in mathematical finance for the pricing of exotic options and is an avenue of present research.

References

[1] F. Müller-Hoissen, "Introduction to noncommutative geometry of commutative algebras and applications in physics," in *Proceedings of the 2nd Mexican School on Gravitation and Mathematical Physics*, (Konstanz), Science Network Publishing, 1998. (Available online at http://kaluza.physik.uni-konstanz.de/2MS/mh/mh.html).

[2] A. Dimakis and F. Müller-Hoissen, "Stochastic differential calculus, the moyal $*$-product, and noncommutative geometry," *Lett. Math. Phys.*, vol. 28, pp. 123–137, 1993.

[3] P. P. Carr and A. Bandyopadhyay, "How to derive the black-scholes equation correctly?," http://www.staff.uiuc.edu/∼bandyopa.

[4] A. Dimakis and F. Müller-Hoissen, "Discrete riemannian geometry," *J. Math. Phys.*, vol. 40, pp. 1518–1548, 1999.